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LETTER TO THE EDITOR

# Partial dynamical symmetry

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**Abstract.** A novel type of symmetry structure, which we call 'partial dynamical symmetry', is discussed. A general algorithm is presented to construct Hamiltonians with such symmetry, for any semisimple group. These Hamiltonians are not invariant under that group, and various irreducible representations are mixed in their eigenstates. However, they possess a subset of eigenstates which do have good symmetry and can therefore be labelled by the irreducible representations of that group. The eigenvalues and wavefunctions of these states are given in closed form. An example of a Hamiltonian with a partial SU(3) symmetry is provided.

Symmetry plays an important role in solving for the eigenstates of the Hamiltonian. In a basis labelled by the irreducible representations (irreps) of the symmetry group, the Hamiltonian matrix admits a block structure such that inequivalent irreps do not mix. Furthermore, eigenstates which belong to the same irrep of the symmetry group are degenerate. Another symmetry concept which has been used extensively in a variety of problems in physics is that of a dynamical symmetry [1–3]. It is a situation in which the Hamiltonian is written in terms of the Casimir invariants of a string of subgroups

$$G_0 \supset \ldots \supset G \supset \ldots \supset G'. \tag{1}$$

A dynamical symmetry is characterized by the following features: (i) All eigenstates can be classified according to the irreps of the groups in the chain. (ii) The wavefunctions, eigenvalues and other observables (e.g. transitions rates) are known analytically. (iii) The wavefunctions do not depend on the Hamiltonian's parameters. Only the last group in the chain, G′, is a symmetry group of the Hamiltonian. An intermediate group in the chain, say G, does not leave the Hamiltonian invariant, so that eigenstates which belong to an irrep of G are usually not degenerate. However, in a dynamical symmetry the Hamiltonian commutes with the Casimir invariants of the groups in the chain, so that states which belong to different irreps are not mixed. The Hamiltonian still has a block structure in a basis characterized by the irreps of the groups.

In applications of group theoretical methods to realistic systems one often finds that the assumed symmetry is only approximate and is fulfilled by only some of the states but not by others. In this letter we explore a particular type of symmetry breaking which is a generalization of the concept of a dynamical symmetry. It is a situation in which only some of the eigenstates of the Hamiltonian exhibit the previously mentioned

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properties (i)–(iii). We refer to such a symmetry structure as 'partial dynamical symmetry'. A Hamiltonian with the above property is not invariant under the group G, nor does it commute with the Casimir invariants of G, so that various irreps are in general mixed in its eigenstates. However, there is a subset of eigenstates for which no such mixing occurs. These states can still be labelled by irreps of G, and their eigenvalues and wavefunctions are known analytically. A case of an approximate partial dynamical symmetry was noticed in the problem of the hydrogen atom in a magnetic field. A dynamical symmetry that exists for weak fields [4], is broken at strong fields except for the quasi-Landau levels [5]. A Hamiltonian with partial SU(3) symmetry and others appears in the study [6] of intrinsic structure in the interacting boson model of nuclei [1]. The purpose of this letter is to raise this empirical phenomeon which occurs naturally in the above systems to the level of an exact, general and novel concept. Furthermore, we shall present a general algorithm for constructing Hamiltonians with partial dynamical symmetry G, for any semisimple group G, which still have G′ (G ⊃ G′) as their symmetry group. For simplicity we take G′ ≡ O(3). We shall demonstrate our general procedure for a Hamiltonian with a partial SU(3) symmetry. We note that, though we are discussing the breaking of a dynamical symmetry, the construction presented below can also be used to generalize the concept of symmetry to that of a partial symmetry.

A semisimple algebra G (of rank $l$) can be described [7] in terms of its Cartan basis composed of maximally commuting subset of generators $H_i$ ($i = 1, \ldots, l$) and ladder operators $E_\alpha$, for any root $\alpha = (\alpha_1, \ldots, \alpha_l)$ defined by $[H_i, E_\alpha] = \alpha_i E_\alpha$. An irrep of G can be described in a basis of common eigenstates $|\lambda\rangle$ of the $H_i$ with weights $\lambda = (\lambda_1, \ldots, \lambda_l)$, ($H_i|\lambda\rangle = \lambda_i|\lambda\rangle$). The highest weight $\Lambda$ is used to characterize the corresponding irrep. The representation which is conjugate to $[\Lambda]$ is denoted by $[\Lambda]^*$. Its weights are obtained by reversing the sign of the weights of $[\Lambda]$, so that $[\Lambda]^*$ can be characterized by its lowest weight $-\Lambda$.

The building blocks of our construction are:

(i) A vacuum state $|\mathrm{vac}\rangle$, which is assumed to be a lowest weight vector $-\Lambda_0$ in the irrep $[\Lambda_0]^*$ of G. It therefore satisfies

$$E_{-\alpha}|\mathrm{vac}\rangle = 0 \qquad \text{for all positive roots } \alpha. \tag{2}$$

(ii) The components $T_{[\Lambda]\lambda}$ of an irreducible tensor operator under the group G belonging to the representation $[\Lambda]$. Henceforth, to keep the notation simple, we shall denote $T_{[\Lambda]\lambda}$ by $T_\lambda$. The highest weight component of $T$ is then $T_\Lambda$. Note that $T^\dagger_\lambda$ transform according to the irrep $[\Lambda]^*$, and that $T^\dagger_\Lambda$ is a lowest weight component in that representation satisfying

$$[E_{-\alpha}, T^\dagger_\Lambda] = 0 \qquad \text{for all } \alpha > 0. \tag{3}$$

Our algorithm is based on requiring the vacuum $|\mathrm{vac}\rangle$, and the highest weight component $T_\Lambda$, to satisfy the following conditions

$$\begin{aligned} &T_\Lambda|\mathrm{vac}\rangle = 0 \qquad [T_\Lambda, T^\dagger_\Lambda]|\mathrm{vac}\rangle = a|\mathrm{vac}\rangle \\ &[[T_\Lambda, T^\dagger_\Lambda], T^\dagger_\Lambda] = bT^\dagger_\Lambda \end{aligned} \tag{4}$$

where $a$, $b$ are constants. In the example given below we shall see that it is possible to satisfy such conditions.

Using the properties of the algebra and of the tensor operator it is possible to show that the following relations result from (4):

$$[E_{-\alpha}, T_\Lambda]|\mathrm{vac}\rangle = 0 \qquad [[E_{-\alpha}, T_\Lambda], T_\Lambda^\dagger]|\mathrm{vac}\rangle = 0$$
$$[[[E_{-\alpha}, T_\Lambda], T_\Lambda^\dagger], T_\Lambda^\dagger] = 0 \tag{5}$$

for any positive root $\alpha$. To get the first relation we use (2), while the other two relations are obtained by using the Jacobi identity once and twice respectively, together with properties (2) and (3). Equations (5) are similarily satisfied when $[E_{-\alpha}, T_\Lambda]$ is replaced by $[E_{-\gamma}, \ldots, [E_{-\beta}, [E_{-\alpha}, T_\Lambda]] \ldots]$ for any positive roots $\alpha, \beta, \ldots, \gamma$. Since any weight vector can be obtained from the highest one by a linear combination of such repetitive applications, it follows that

$$T_\lambda|\mathrm{vac}\rangle = 0 \qquad [T_\lambda, T_\Lambda^\dagger]|\mathrm{vac}\rangle = 0$$
$$[[T_\lambda, T_\Lambda^\dagger], T_\Lambda^\dagger] = 0 \tag{6}$$

for all $\lambda \neq \Lambda$.

Equations (4) and (6) can be viewed as a generalization of the conditions satisfied by harmonic oscillators operators. Note, however, that the first two relations in (4) and (6) are not an operator identity and that no further assumption is made on the vacuum state nor on the operator $T_\Lambda$, which may be composite objects.

Consider the sequence of states

$$|k\rangle \propto (T_\Lambda^\dagger)^k|\mathrm{vac}\rangle. \tag{7}$$

Due to (4) these are eigenstates of $T_\Lambda^\dagger T_\Lambda$, and because of (6) they have the property that $T_\lambda|k\rangle = 0$ for any $\lambda \neq \Lambda$. We conclude that $|k\rangle$ are exact eigenstates of any Hamiltonian of the form

$$H = h_{\Lambda\Lambda} T_\Lambda^\dagger T_\Lambda + \sum_{\lambda,\sigma \neq \Lambda} h_{\lambda\sigma} T_\lambda^\dagger T_\sigma \tag{8}$$

with eigenvalues

$$E_k = h_{\Lambda\Lambda}[ka + \tfrac{1}{2}k(k-1)b] \tag{9}$$

that are independent of the parameters $h_{\lambda\sigma}$. It is straightforward to show that $|k\rangle$ is the lowest weight vector in the representation $[\Lambda_k]^* \equiv [\Lambda_0 + k\Lambda]^*$ of G. Since for a general set of coefficients $h_{\lambda\sigma}$, the Hamiltonian (8) is not a G scalar, all of its eigenstates, except $|k\rangle$, will be a mixture of irreps of G. We have thus accomplished our initial goal, except that the Hamiltonian (8) is generally not rotational (or G') invariant. To remedy that, we decompose the tensor operator $T_\lambda$ into rotational tensor operators $T_{LM}$ of good angular momentum $L$ and projection $M$. We can then construct rotational scalar Hamiltonians

$$H = \sum_{LM} h_L T_{LM}^\dagger T_{LM} \tag{10}$$

where we have omitted possible multiplicity indices of the representations. If in addition we assume that the highest weight vector $\Lambda$ has a well defined, multiplicity-free, $\bar{L}$-value in the decomposition to O(3), then (10) is of the form (8) with $h_{\Lambda\Lambda} = h_{\bar{L}}$. $|k\rangle$ are therefore eigenstates of the scalar Hamiltonian (10). When $\bar{L} \neq 0$ and/or $|\mathrm{vac}\rangle$ does not have zero angular momentum, the states $|k\rangle$ do not have good $L$. The O(3) symmetry is thus spontaneously broken in the sense that the eigenstates $|k\rangle$ do not have the same symmetry as the Hamiltonian. States of good angular momentum can be obtained by projection from $|k\rangle$. Since $H$ is a scalar, the projection operator commutes with it, so that the projected states $|[\Lambda_k]L, M\rangle$ are also eigenstates of (10). A special case for (10) may be a Casimir invariant of G for which a dynamical symmetry occurs. However, with arbitrary coefficients $h_L$, a generic Hamiltonian in (10) is not a G scalar, yet a subset of its eigenstates $|[\Lambda_k]L, M\rangle$ are characterized by good quantum numbers of $\mathrm{G} \supset \mathrm{O(3)}$. It should be noted that these special states belong to particular irreps of G (of the form $[\Lambda_0 + k\Lambda]^*$), and in general they span only part of these irreps, i.e. other states, originally in these special irreps, no longer have good G symmetry. We also note that the above discussion holds for any compact group G′ (replacing O(3)), since appropriate projection operators can be constructed in such a case.

To illustrate the general formalism, we use the interacting boson model [1], widely used for the description of collective states in nuclei. The building blocks of the model are one monopole boson ($s^\dagger$) and five quadrupole bosons ($d^\dagger_\mu$). The bilinear combinations of one creation operator and one annihilation operator generate a $U(6)$ algebra. The Hamiltonian contains scalar interactions which conserve the total number of bosons $N$. One of the possible dynamical symmetries of the model corresponds to the chain [1] $\mathrm{U(6)} \supset \mathrm{SU(3)} \supset \mathrm{O(3)}$ which describes rotational nuclei. Thus the groups G and G′ of equation (1) are SU(3) and O(3), respectively. We now construct Hamiltonians with partial SU(3) dynamical symmetry, following the general algorithm presented above.

SU(3) is a semisimple group of rank two [8]. The vacuum state $|\mathrm{vac}\rangle$ is taken to be a condensate of $N$ bosons of the form $|c\rangle = (N!)^{-1/2}[(s^\dagger + \sqrt{2}\, d^\dagger_0)/\sqrt{3}]^N|0\rangle$, where $|0\rangle$ is the boson vacuum of the model (no bosons). The condensate $|c\rangle$ is a lowest weight state [9] in the $(2N, 0)$ irrep of SU(3) and is thus annihilated by the ladder operators [8] $T_-$, $U_-$ and $V_+$ which correspond to the negative roots (equation (2)). The desired SU(3) tensor operator $T^\dagger_\lambda$ is constructed from boson pair operators of angular momentum $L = 0$ and 2

$$P^\dagger_0 = d^\dagger \cdot d^\dagger - 2(s^\dagger)^2 \qquad P^\dagger_{2\mu} = \sqrt{2}\, s^\dagger d^\dagger_\mu + \sqrt{\tfrac{7}{2}}\,(d^\dagger d^\dagger)^{(2)}_\mu. \tag{11}$$

They transform under SU(3) as the $(0, 2)$ irrep. The operators $P_0$ and $P_{2\mu}$ transform like the irrep $(2, 0)$ and correspond to the $T_\lambda$. The highest weight component is $T_\Lambda \equiv P_{2,2}$. A straightforward calculation shows that $P_{2,2}$ and the condensate $|c\rangle$ satisfy the basic conditions (4) for any $N$ with $a = 6N + 9$ and $b = 12$. It then follows that relations (6) are also satisfied, where $T_\lambda$ is replaced by $P_0$ or $P_{2\mu}$ ($\mu \neq 2$). In analogy with (10) we consider the following O(3) scalar Hamiltonian

$$H = h_0 P^\dagger_0 P_0 + h_2 \sum_\mu P^\dagger_{2\mu} P_{2\mu} \tag{12}$$

where $h_0$, $h_2$ are arbitrary constants. This Hamiltonian appeared previously [6] as an intrinsic Hamiltonian for which the condensate $|c\rangle$ is an exact eigenstate. For $h_2 = 2h_0$, $H$ is an SU(3) scalar related to its quadratic Casimir invariant, while for $h_2 = -2h_0/5$, $H$ is a $(2, 2)$ tensor component. Thus, for arbitrary $h_0$ and $h_2$, $H$ is not an SU(3) scalar. Nevertheless, an SU(3) partial symmetry exists, since the sequence of states (see equation (7))

$$|k\rangle \propto (P^\dagger_{2,2})^k|c\rangle \tag{13}$$

are eigenstates of (12) and continue to have good SU(3) symmetry. Their eigenvalues are independent of $h_0$ and are given by (9) with $h_{\Lambda\Lambda} = h_2$.

In order to preserve the total boson number $N$, we can take in (13) a condensate with $N-2k$ bosons. In doing so we obtain a set of deformed states [9] which are lowest weight states in the SU(3) irreps $(2N-4k, 2k)$ with $2k \leq N$. In the nuclear physics terminology [8, 9] they are referred to as 'intrinsic states' associated with rotational bands of an axially deformed nucleus. They have well defined angular momentum projection ($K=2k$) along the symmetry axis and represent $\gamma^k$ bands. (Here and in what follows, $\beta$ and $\gamma$ denote the quadrupole deformation parameters.) In particular, $|k=0\rangle$ represents the ground-state band ($K=0$) and $|k=1\rangle$ is the $\gamma$-band ($K=2$). The intrinsic states break the O(3) symmetry. Eigenstates of (12) with good angular momentum $L \geq K$ (and good SU(3) symmetry), $|(2N-4k, 2k)\ K=2k;\ L, M\rangle$, are obtained by angular momentum projection.

An example of this SU(3) partial dynamical symmetry is displayed in figure 1 for $N=7$ bosons. Typical spectra of the Hamiltonian (12) are shown for two cases: (a) $h_2=2h_0$ and (b) $h_2=1.25h_0$. Case (a) corresponds to a full SU(3) symmetry and all states are arranged in degenerate SU(3) multiplets. For case (b) the SU(3) symmetry is broken, so that most of the eigenstates have a mixture of SU(3) irreps and the above SU(3) degeneracy is lifted. However, some of the eigenstates (marked by a + in figure 1) continue to carry good SU(3) $\supset$ O(3) representation labels and are arranged in multiplets. Within each such multiplet the $L$ degeneracy can be lifted by adding the Casimir invariant of O(3) ($L^2$) to the Hamiltonian (12), contributing an $L(L+1)$ splitting. The SU(3) symmetry of case (a) then becomes an SU(3) dynamical symmetry, with rotational bands built on each of the SU(3) irreps. In case (b) we obtain an SU(3) partial dynamical symmetry. The ground-state $K=0$ band continues to carry good SU(3) labels (14, 0). The $\beta(K=0)$ and $\gamma(K=2)$ bands, which originally were degenerate in the SU(3) dynamical symmetry limit (and belonging to the (10, 2) representation), split. Only the members of the $\gamma$-band continue to carry good SU(3) labels (10, 2), while the $\beta$-band has a mixture of several SU(3) irreps. Similarly, the $\gamma^2(K=4)$ band and the $\gamma^3(K=6)$ bands preserve their SU(3) character. All other states exhibit mixing.

To further visualize the phenomena of partial dynamical symmetry, we show in figure 2 the SU(3) content of three eigenstates of case (b). These are the seventh and

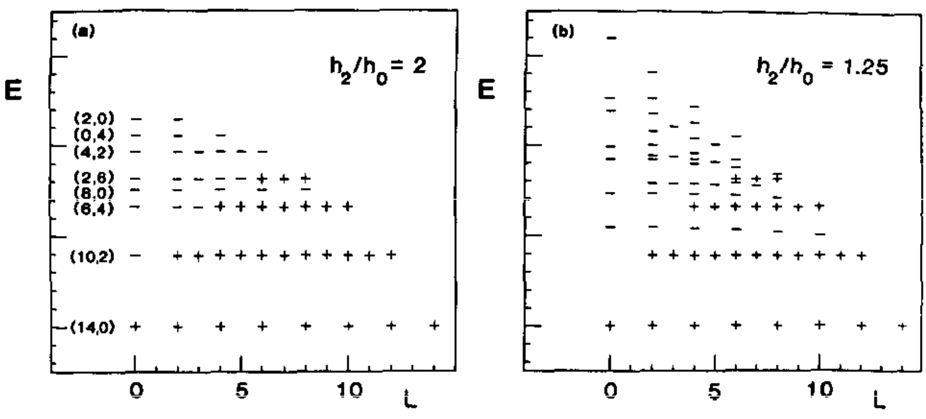


**Figure 1.** Spectra (energy $E$ versus angular momentum $L$) of the Hamiltonian (12). ($a$) For $h_2/h_0=2$. SU(3) symmetry labels $(\lambda, \mu)$ are shown on the left. Some levels with $L \neq 0$ exhibit multiplicity which is not shown. ($b$) For $h_2/h_0=1.25$. Levels which continue to exhibit SU(3) symmetry are marked by a + symbol. The energy scale is arbitrary and the same value of $h_2$ was used in both cases.

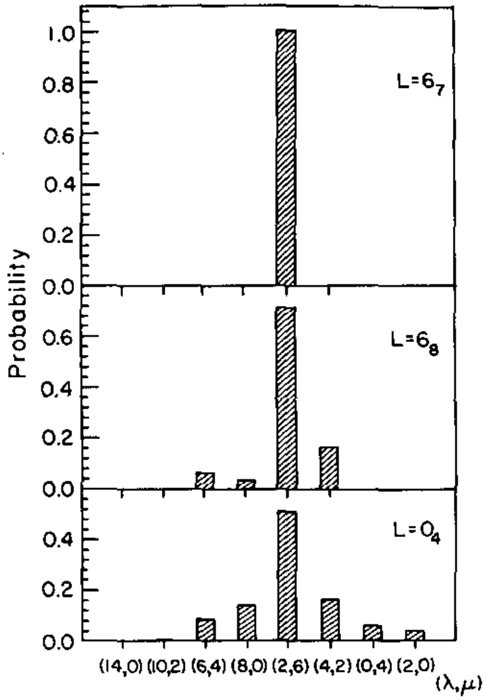


**Figure 2.** SU(3) decomposition (probability versus SU(3) irrep $(\lambda, \mu)$) for selected states in the spectrum of figure 1($b$).

eighth $L=6$ states and the fourth $L=0$ state, all of which have a dominant $(2,6)$ component. The state $L=6_7$ is one of the solvable states in the $\gamma^3(K=6)$ band and is 100% in the $(2,6)$ irrep. The state $L=6_8$ which in case (a) was a pure $(2,6)$ state, degenerate with $L=6_7$, is now a mixture of several SU(3) representations and only 73% $(2,6)$. Other states are even more strongly mixed; the fifth $L=0$ state in case (a) which belonged to the irrep $(2,6)$, had crossed the fourth $L=0$ level to become the $L=0_4$ level of case (b). It has only 51% in the irrep $(2,6)$ and exhibits a significant spread over six SU(3) irreps.

To conclude, we have presented an algorithm to construct Hamiltonians which possess partial dynamical symmetry when the Hamiltonian does not have good symmetry, but a subset of its eigenstates do. One of the striking features of such Hamiltonians is the possibility that these states span only part of the corresponding irreps. This is not the case in the quasi-exactly solvable Hamiltonians constructed recently [10], in which the solvable states form complete representations. The coexistence of solvable and unsolvable states, together with the availability of an algorithm, distinguish the notion of partial dynamical symmetry from the notion of accidental degeneracy [11], where all levels are arranged in degenerate multiplets. Additional examples of partial dynamical symmetry will be given in a longer publication [12]. The concept of partial dynamical symmetry may play an important role in exploring systems which exhibit both regular and irregular behaviour, and in constructing models of complex systems in which some of the elementary excitations are described by solvable states.

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